# How Far Can We Go Beyond Linear Cryptanalysis?

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# Outline

1. Introduction
2. Optimal distinguisher between two random sources
   - General case
   - One source following a uniform distribution
   - Source of random bit strings
   - Statistical distinguishers
3. Optimal distinguisher between two random oracles
   - Beyond linear probabilities and linear expressions
   - Beyond the piling-up lemma
   - From distinguishers to key-recovery attacks
4. Conclusion

# Introduction

## Original Motivation

To give a generalization of linear cryptanalysis.

## Result

The paper turns out to propose a very general statistical framework which can be used to construct and study optimal distinguishers, and to generalize the fundamental concepts behind linear cryptanalysis.

# Previous Work

The original linear cryptanalysis was proposed by Matsui at EUROCRYPT'93. Since then, several generalizations have been proposed.

- Kaliski and Robshaw used multiple linear approximations,
- Vaudenay proposed the $\chi^2$ attack, where a cipher can simply be considered as a black box,
- Harpes, Kramer, and Massey replaced linear expressions with I/O sums,
- Harpes and Massey considered partition pairs of the input and output spaces of the cipher,
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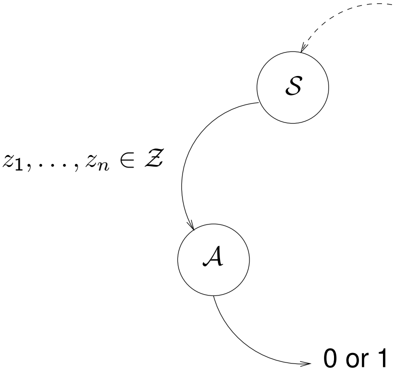

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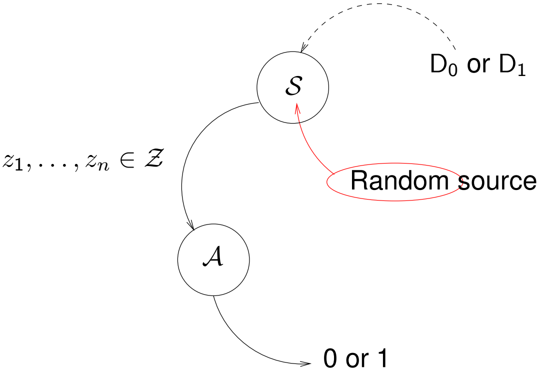

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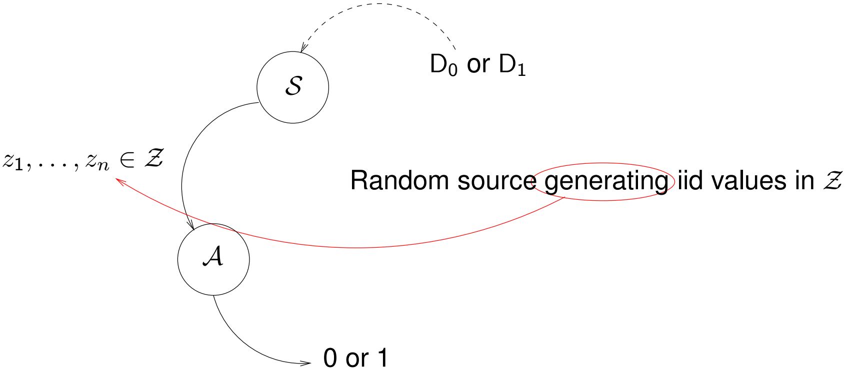

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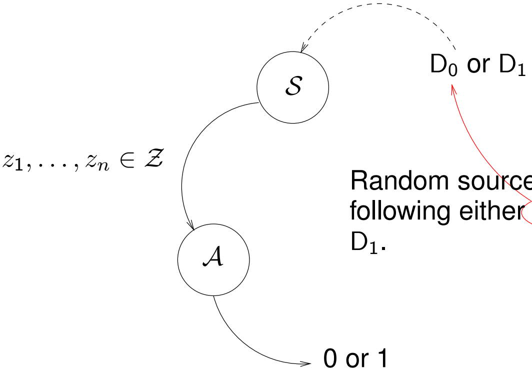


Random source generating iid values in $\mathcal{Z}$ following either distribution $D_0$ or $D_1$.

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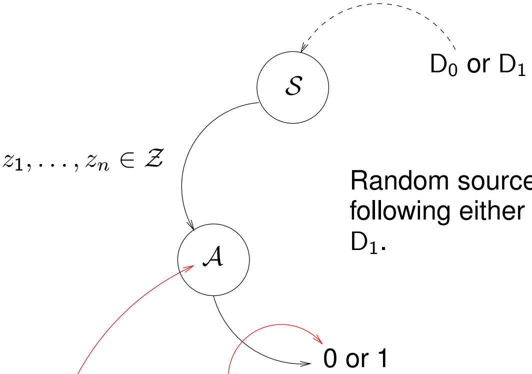


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The distinguisher decides whether $D = D_0$ or $D = D_1$

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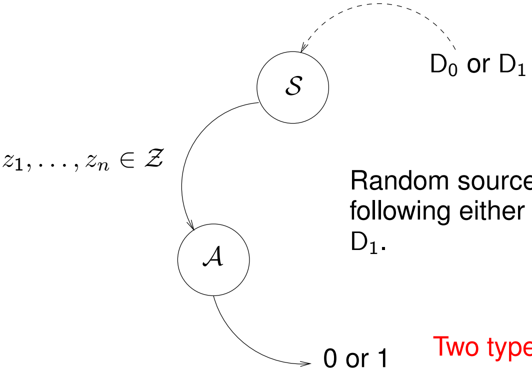


Random source generating iid values in $\mathcal{Z}$ following either distribution $\mathsf{D}_0$ or $\mathsf{D}_1$.

Two types of mistake:

The distinguisher decides whether $\mathsf{D} = \mathsf{D}_0$ or $\mathsf{D} = \mathsf{D}_1$

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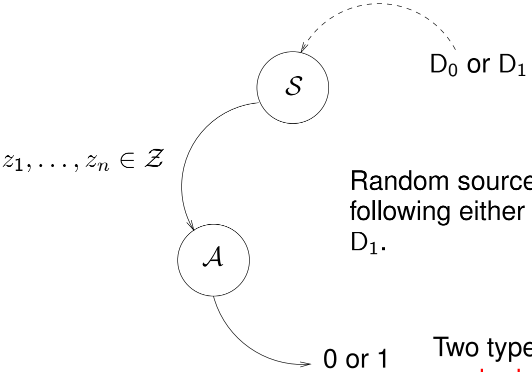


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Two types of mistake:
- output 0 when $\mathsf{D} = \mathsf{D}_1$ (probability $\alpha$)
- output 1 when $\mathsf{D} = \mathsf{D}_0$ (probability $\beta$)

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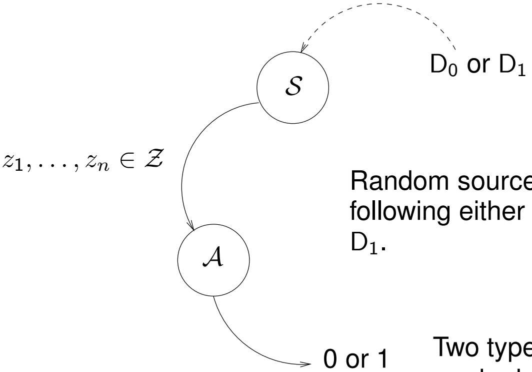
Optimal distinguisher $\Leftrightarrow \mathsf{P_e} = \frac{1}{2}(\alpha + \beta)$ minimum




Random source generating iid values in $\mathcal{Z}$ following either distribution D$_0$ or D$_1$.

The distinguisher decides whether D $=$ D$_0$ or D $=$ D$_1$

Two types of mistake:
- output 0 when D $=$ D$_1$ (probability $\alpha$)
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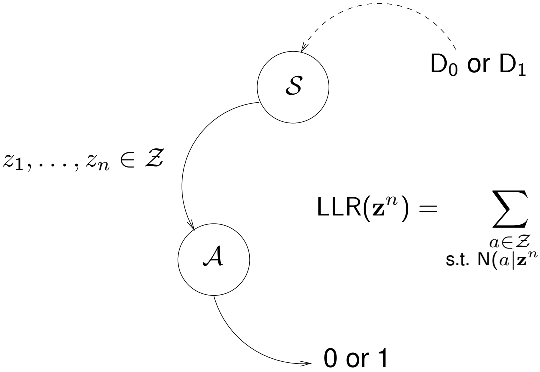
# General case (2)




$$\mathrm{LLR}(\mathbf{z}^n) = \sum_{\substack{a \in \mathcal{Z} \\ \text{s.t. } \mathrm{N}(a|\mathbf{z}^n)>0}} \mathrm{N}(a \mid \mathbf{z}^n) \log \frac{\Pr_{\mathrm{D}_0}[a]}{\Pr_{\mathrm{D}_1}[a]}$$

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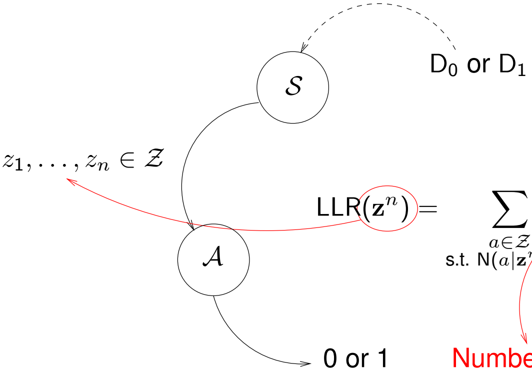


Probability that $\mathcal{S}$ sent $a$ when $\mathrm{D} = \mathrm{D}_0$

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Number of times $\mathcal{S}$ sent $a$

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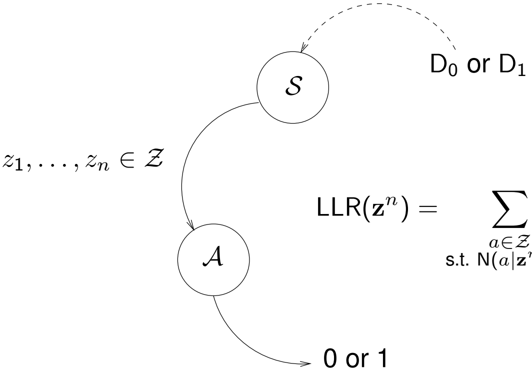


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Optimal Rule:
choose 0 when $\mathsf{LLR}(\mathbf{z}^n) \geq 0$
choose 1 when $\mathsf{LLR}(\mathbf{z}^n) < 0$

This minimizes $P_e \Rightarrow$ optimal distinguisher
(aka Neyman-Pearson lemma)

# General case (3)

For a given $\mathsf{P_e}$, how many queries does the distinguisher need?

## Theorem

Considering that

- $Z_1, \ldots, Z_n$ are iid, following distribution $\mathsf{D} \in \{\mathsf{D_0}, \mathsf{D_1}\}$,
- $\mathsf{D_0}$ is close to $\mathsf{D_1}$, i.e., $\Pr_{\mathsf{D_0}}[z] - \Pr_{\mathsf{D_1}}[z] = \epsilon_z \ll 1$,

$$n = \frac{d}{\sum_{z \in \mathcal{Z}} \frac{\epsilon_z^2}{p_z}} \qquad \text{with} \qquad \mathsf{P_e} \approx 1 - \Phi\left(\frac{\sqrt{d}}{2}\right).$$

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# Outline

# One source following a uniform distribution

## Squared Euclidean Imbalance (SEI)

If $\mathsf{D}_1$ is the uniform distribution (i.e., $\Pr_{\mathsf{D}_1}[z] = p_z = \frac{1}{|\mathcal{Z}|}$), we define the Squared Euclidean Imbalance (SEI):

$$\Delta(\mathsf{D}_0) = |\mathcal{Z}| \sum_{z \in \mathcal{Z}} \epsilon_z^2 \quad .$$

## Corollary

$$n = \frac{d}{\Delta(\mathsf{D}_0)} \qquad \text{with} \qquad \mathsf{P_e} \approx 1 - \Phi\left(\frac{\sqrt{d}}{2}\right) \quad .$$

$\Rightarrow$ The complexity of distinguishing $\mathsf{D}_0$ from $\mathsf{D}_1$ can be measured by means of the SEI.

# Link to $\chi^2$ attacks

In a $\chi^2$ cryptanalysis, the adversary does not need to know $D_1$, i.e., what exactly happens in the inner transformations of the cipher (which can therefore be considered as a *black box*).

- Complexity of a $\chi^2$ attack $\rightarrow O(1/\Delta(D_0))$
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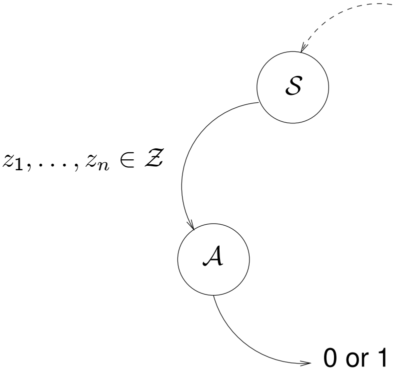


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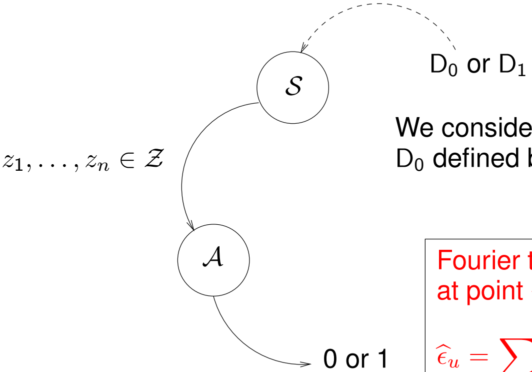


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Fourier transform of D$_0$ at point $u \in \mathcal{Z}$ :

$$\widehat{\epsilon}_u = \sum_{z\in\mathcal{Z}} (-1)^{u\cdot z} \epsilon_z$$

# Source of random bit strings (2)

Properties of the SEI (shown using the Fourier transform):

- $\Delta(\mathsf{D}_0) = \sum_{u \in \mathcal{Z}} \widehat{\epsilon}_u^{\,2}$
- When $B$ is a random bit, recall the linear probability is $\mathsf{LP}(B) = (2\Pr[B = 0] - 1)^2$. Then,
    - $\Delta(\mathsf{D}_0) = \sum_{w \in \mathcal{Z} \setminus \{0\}} \mathsf{LP}(w \cdot \mathcal{Z})$
    - with $\mathsf{LP}^{\mathcal{Z}}_{\max} = \max_{w \in \mathcal{Z} \setminus \{0\}} \mathsf{LP}(w \cdot \mathcal{Z})$,

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# Outline

# Statistical distinguishers

We know how to distinguish distributions in $\{0,1\}^\ell$ of *small cardinality* (i.e., $\ell$ is small).

What if the source generates variables in $\{0,1\}^L$ where $L$ is *large*?

Solution:

- *reduce the sample space* by means of a projection:

$$h : \{0,1\}^L \longrightarrow \mathcal{Z} \quad .$$

- $Z = h(S) \in \mathcal{Z}$ follows either $D_0$ or $D_1$.

But how should we choose the projection $h$?!? (This may be where cryptanalysis becomes an art !)

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# First example of a statistical distinguisher

For some non-zero $a \in \{0,1\}^L$

$$\begin{array}{rccc} h: & \{0,1\}^L & \longrightarrow & \mathcal{Z} = \{0,1\} \\ & S & \longmapsto & h(S) = a \cdot S \end{array} .$$

This is a linear distinguisher.

We note that $\Delta(h(S)) = \mathrm{LP}(a \cdot S) \leq \mathrm{LP}^S_{\max}$.

Modern ciphers have a bounded $\mathrm{LP}^S_{\max}$
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where $h$ is GF (2)-linear.

**Theorem**

$$\Delta(h(S)) \leq (2^\ell - 1)\mathrm{LP}^S_{\max} \ .$$

Ciphers protected against linear cryptanalysis (bounded $\mathrm{LP}^S_{\max}$)
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Ciphers protected against linear cryptanalysis (bounded $\mathsf{LP}^S_{\mathsf{max}}$)
$\Rightarrow$ somewhat protected against several generalizations!

# Bounded $\mathrm{LP}^S_{\max}$ and low advantage are not equivalent!

Is it possible to find a distinguisher

- with a high advantage,
- even though the value of $\mathrm{LP}^S_{\max}$ is small?

Practical examples exist. For example

- Jakobsen and Knudsen's interpolation attack (where quadratic functions are used),
- Courtois' bi-linear cryptanalysis.

In the paper we provide an example of a source

- impossible to break with a linear distinguisher,
- trivially broken by a (well-chosen) non-linear distinguisher.

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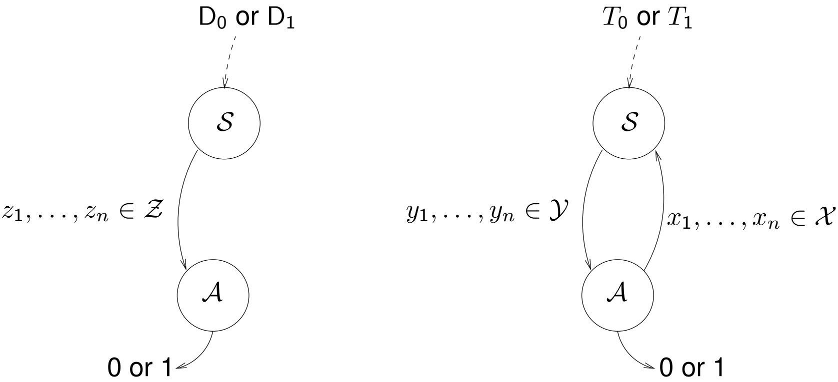
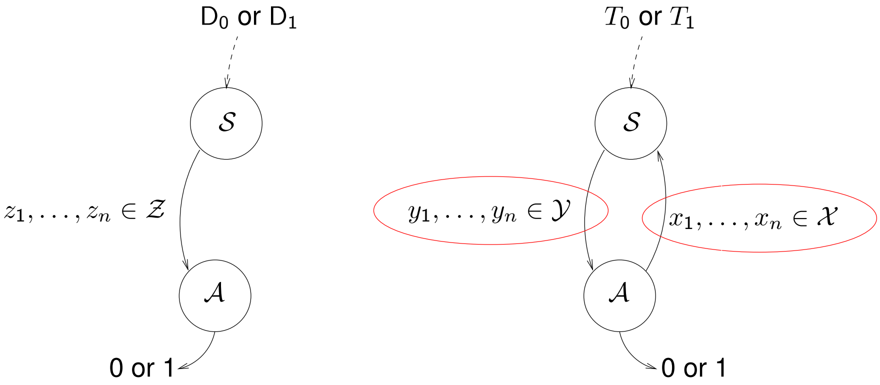
# Outline

# Beyond linear probabilities and linear expressions (1)



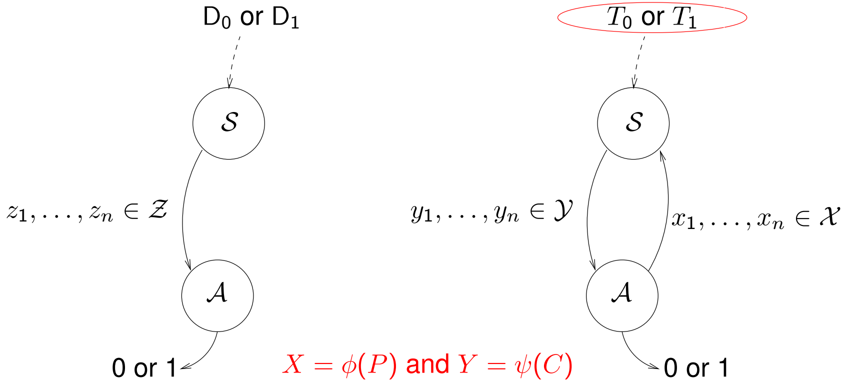

We know how to distinguish random sources.
$\rightarrow$ what about random oracles?

# Beyond linear probabilities and linear expressions (1)




$Z \in \mathcal{Z}$ becomes a couple of random variables $(X, Y) \in \mathcal{X} \times \mathcal{Y}$.

# Beyond linear probabilities and linear expressions (1)




$X = \phi(P)$ and $Y = \psi(C)$

known plaintext attack $\rightarrow P \sim$ uniform distrib. $\rightarrow X \sim$ uniform distrib.

Distribution of $Y$ defined by a transition matrix:

$[T]_{x,y} = \Pr[Y = y \mid X = x]$

# Beyond linear probabilities and linear expressions (2)

## Transition Matrix

$$[T]_{x,y} = \mathrm{Pr}\left[Y = y \mid X = x\right] .$$

When $T = T_1$, $Y$ is uniformly distributed.

## Bias Matrix

$$B = T_0 - T_1 .$$

## Link between bias matrix and SEI

$$\Delta(\mathrm{D}_0) = \frac{|\mathcal{Y}|}{|\mathcal{X}|} \parallel B \parallel_2^2 .$$

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# Piling-up transition matrices

$$
\begin{tikzcd}
P^{(1)} \arrow[r, "\phi"] \arrow[d, "\mathsf{C}^{(1)}"'] & X \arrow[d, "T^{(1)}"] \\
P^{(2)} \arrow[r, "\xi"] \arrow[d, "\mathsf{C}^{(2)}"'] & W \arrow[d, "T^{(2)}"] \\
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If $X \leftrightarrow W \leftrightarrow Y$ is a Markov chain

$$T = T^{(1)} \times T^{(2)}$$

# Piling-up transition matrices

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\begin{tikzcd}
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If $X \leftrightarrow W \leftrightarrow Y$ is a Markov chain

$$T = T^{(1)} \times T^{(2)} \rightsquigarrow B = B^{(1)} \times B^{(2)} \rightsquigarrow \| B \|_2 \leq \| B^{(1)} \|_2 \times \| B^{(2)} \|_2$$

# Outline

1. Introduction
2. Optimal distinguisher between two random sources
   - General case
   - One source following a uniform distribution
   - Source of random bit strings
   - Statistical distinguishers
3. Optimal distinguisher between two random oracles
   - Beyond linear probabilities and linear expressions
   - Beyond the piling-up lemma
   - From distinguishers to key-recovery attacks
4. Conclusion

# Key recovery attacks

The framework can be adapted to key recovery.

In the paper we show how to build an optimal key ranking procedure that recovers a $k$ bits key provided that the number of samples $n$ is s.t.

$$n \geq \frac{4k \log 2}{\Delta(\mathrm{D}_0)} .$$

This formula was used to estimate the complexity of attacks against E0 (don't miss this morning's last talk!!).

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- We defined a rigorous statistical framework in order to interpret LC and its generalizations in a unified way.
- Modern block ciphers are proven resistant against LC.
- This resistance extends to linear generalizations of LC,
- …but definitely not to non-linear ones!

Thank you for your attention!

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